# Existence of equivariant biharmonic maps

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**Abstract**

We consider two compact Riemannian manifolds $M$ and $N$ and a compact Lie group $G$ that acts on both by isometries. Under certain assumptions on the structure of $M$ and of the quotient space $M/G$, we construct equivariant biharmonic maps $u : M \to N$ with prescribed boundary data.

## 1 Introduction

Let $(M, g)$ be an $m$-dimensional, smooth, compact Riemannian manifold with boundary, where $m = 3$ or $m = 4$, and let $(N, \langle \cdot, \cdot \rangle)$ be a smooth, compact Riemannian manifold without boundary. We assume that there exists a compact Lie group $G$ acting smoothly on both $M$ and $N$ by isometries. We study maps $u : M \to N$ that are equivariant with respect to these actions in the sense that

$$u(ax) = au(x)$$

for all $a \in G$ and $x \in M$. More precisely, we want to find equivariant maps that are also biharmonic in the following sense. For a smooth map $u : M \to N$, we have a covariant derivative $D$ on the pull-back vector bundle $u^{-1}TN$ over $M$ induced by the Levi-Civita connection on $N$, and we have a corresponding covariant derivative on $T^*M \otimes u^{-1}TN$, denoted by $D$ as well. This gives rise to the section $Ddu$ of $T^*M \otimes T^*M \otimes u^{-1}TN$, the trace of which is called the tension field of $u$ and denoted by

$$\tau(u) = \operatorname{tr} Ddu.$$

A biharmonic map is a critical point of the functional

$$E_2(u) = \int_M |\tau(u)|^2 \, d\text{vol}.$$

This variational problem gives rise to the Euler-Lagrange equation

$$\Delta\tau(u) + \operatorname{tr} R(\tau(u), du)du = 0, \tag{1}$$

where $\Delta$ is the Laplacian belonging to $D$ and $R$ is the Riemann curvature tensor on $N$ (pulled back to $u^{-1}TN$).

Despite its variational nature, the problem is rather challenging from the analysis point of view and there are no general existence results, except under the assumption that $N$ is a homogeneous space [17]. (There are also some non-existence results under the assumption that $N$ has non-positive sectional curvature [10, 2].) In this paper, we study the question for a different kind of symmetry, restricting our attention to equivariant maps.

For $x \in M$, let $Gx$ be the orbit of $x$ under the group action. Note that every orbit is an embedded submanifold of $M$ [4, Corollary VI.1.3]. In particular, every orbit has a well-defined dimension, and we can decompose $M$ according to these dimensions. We identify each orbit with the corresponding point in the quotient space $M/G$ and denote by $Q^j$ the subset of $M/G$ comprising all $j$-dimensional orbits for $j = 0, \ldots, m$. Furthermore, let

$$M^j = \bigcup_{O \in Q^j} O$$

denote the union of all $j$-dimensional orbits. We will impose some conditions on $M^0$ and $M^{m-3}$ for our main results (i.e., on $M^0$ only if $m = 3$ and on $M^0$ and $M^1$ if $m = 4$). This is in order to take advantage of the symmetry provided by the group action. We will also impose the following condition on the manifold $M$.

**Definition 1.** *We say that $M$ is* dilatable *if there exist a number $c > 0$ and a smooth tangent vector field $X$ on $\overline{M}$ such that at every point $x \in M$ and for every $Y \in T_x M$, the inequality*

$$|Y|^2(\operatorname{div} X - c) \geq 2g(\nabla_Y X(x), Y) \tag{2}$$

*holds true.*

This condition gives a relation between the Lie derivative of the volume form with respect to $X$, which is $\mathcal{L}_X d\mathrm{vol} = \operatorname{div} X \, d\mathrm{vol}$, and the Lie derivative of the metric, which is $\mathcal{L}_X g = g(\nabla X, \cdot) + g(\cdot, \nabla X)$. In other words, it compares the rate at which $X$ generates volume with the rate at which it stretches tangent vectors. The condition is satisfied, e.g., if $M = \overline{\Omega}$ for an open set $\Omega \subset \mathbb{R}^m$ with smooth boundary and if $X(x) = x$ for $x \in \Omega$. The purpose of (2) is to give control of the Dirichlet energy

$$E_1(u) = \frac{1}{2} \int_M |du|^2 \, d\mathrm{vol}$$

in terms of $E_2$ with a Pohozaev type argument (used in the work of the first author [8] and extended by the second author [17]). In the absence of such a condition, the main ideas from this paper will still work for functionals such as $E_2 + aE_1$ for $a > 0$, but we leave it to the reader to work out the details.

If we test (2) with the vectors of a local orthonormal tangent frame field, we see that it implies $\operatorname{div} X \geq mc/(m-2)$. Hence a compact dilatable manifold necessarily has a non-empty boundary.

For the sake of convenience, we assume that $N$ is isometrically embedded in a Euclidean space $\mathbb{R}^{\bar{n}}$, although the theory can also be developed without the use of such an ambient space [9]. By the Nash embedding theorem [19], this assumption does not entail a loss of generality. For $1 \leq p \leq \infty$ and $k \in \mathbb{N}$, let $W^{k,p}(M; N)$ denote the space of all maps $u \in W^{k,p}(M; \mathbb{R}^{\bar{n}})$ such that $u(x) \in N$ for almost every $x \in M$. For $u : M \to N$, let $L^p(u^{-1}TN)$ denote the space of all sections $\Xi$ of $u^{-1}TN$ such that $|\Xi| \in L^p(M)$. Let $A$ denote the second fundamental form of the submanifold $N \subset \mathbb{R}^{\bar{n}}$. Then for any smooth map $u : M \to N$ and any smooth section $\Xi$ of $u^{-1}TN$, we have

$$D\Xi = d\Xi + A(\Xi, du), \tag{3}$$

where the exterior derivative $d$ is applied component-wise. Thus we can write

$$\tau(u) = \Delta u + \operatorname{tr} A(du, du),$$

where $\Delta$ denotes the Laplace-Beltrami operator, with a sign convention that makes it negative semidefinite. Note that $\tau(u)$ is still well-defined as a distribution for all $u \in W^{1,2}(M; N)$. Now let $W^{\tau,2}(M; N)$ denote the space of all $u \in W^{1,2}(M; N)$ such that $\tau(u)$ is represented by an element of $L^2(u^{-1}TN)$. This is a reasonable space to consider when studying the functional $E_2$, but since maps in $W^{\tau,2}(M; N)$ can be rather more irregular than one would normally expect in a Sobolev space, we use a slightly smaller space.

Suppose that we have a map $u \in W^{\tau,2}(M; N)$ such that

$$\int_M \left( \frac{1}{2} |du|^2 \operatorname{div} X - \operatorname{tr} \langle du(\nabla X), du \rangle - \langle du(X), \tau(u) \rangle \right) d\mathrm{vol} = 0 \tag{4}$$

for all smooth tangent vector fields $X$ on $M$ with $\operatorname{supp} X \subset \operatorname{Int} M = M \backslash \partial M$. This condition is automatically satisfied for $u \in W^{2,2}(M; N)$ (and can be verified by an integration by parts) and plays an important part in the regularity theory for harmonic maps [6, 3]. If a map $u \in W^{\tau,2}(M; N)$ satisfies (4), then we can at least obtain some additional control of $du$ away from a small set in $M$. In the literature, this is often done only for maps defined on a domain in $\mathbb{R}^m$ [25, 16], but the arguments can be extended to other manifolds (for example, the crucial monotonicity formula for harmonic maps has been proved in greater generality by Große-Brauckmann [7]).

Since $\partial M$ is non-empty and we work with a second order variational problem, it is natural to impose boundary conditions of the form

$$u = u_0, \quad du = du_0 \quad \text{on } \partial M$$

for a given equivariant map $u_0 : M \to N$. For simplicity, we assume that $u_0$ is smooth. The first of these conditions can be interpreted in the sense of traces for any $u \in W^{1,2}(M; N)$. For $u \in W^{\tau,2}(M; N)$, because we have $\Delta u \in L^1(M; \mathbb{R}^{\bar{n}})$, there is also a natural interpretation of the normal derivative on $\partial M$, while the tangential derivative is automatically fixed by the condition $u = u_0$ on $\partial M$. Thus for $u_0 \in C^\infty(M; N)$, we define $W^{\tau,2}_{u_0}(M; N)$ to be the space of all $u \in W^{\tau,2}(M; N)$ with $u|_{\partial M} = u_0|_{\partial M}$ in the sense of traces and

$$\int_M (\langle du, d\phi \rangle + \langle \Delta u, \phi \rangle) \, d\mathrm{vol} = \int_{\partial M} \langle du_0(\nu), \phi \rangle \, d\sigma$$

for any $\phi \in C^\infty(M; \mathbb{R}^{\bar{n}})$, where $\nu$ is the outer normal vector on $\partial M$ and $d\sigma$ is the surface form on $\partial M$ induced by $g$. If we work with identity (4), then we want to be able to extend it to the boundary. Thus let $K_{u_0}(M; N)$ be the space of all $u \in W^{\tau,2}_{u_0}(M; N)$ such that

$$\int_M \left( \frac{1}{2} |du|^2 \operatorname{div} X - \operatorname{tr} \langle du(\nabla X), du \rangle - \langle du(X), \tau(u) \rangle \right) d\mathrm{vol} = \int_{\partial M} \left( \frac{1}{2} |du_0|^2 g(X, \nu) - \langle du_0(X), du_0(\nu) \rangle \right) d\sigma$$

for any smooth tangent vector field $X$ on $M$. If $U \subset M$ is open, the $K_{u_0}(U;N)$ is defined analogously, allowing only vector fields with $\operatorname{supp} X \subset U$. If $U \cap \partial M = \emptyset$, then we also write $K(U;N)$.

We first have a result on existence of minimisers. Here and subsequently, we write $\mathcal{H}^j$ for the $j$-dimensional Hausdorff measure on $M$.

**Theorem 2.** *Suppose that $M$ is dilatable and $\mathcal{H}^{m-2}(M^0) = \mathcal{H}^{m-2}(M^{m-3}) = 0$. Let $u_0 \in C^\infty(M;N)$ be an equivariant map. Then there exists an equivariant map $u \in K_{u_0}(M;N)$ such that $E_2(u) \leq E_2(v)$ for any equivariant $v \in K_{u_0}(M;N)$.*

In other words, the functional $E_2$ has a minimiser among all equivariant maps in $K_{u_0}(M;N)$ under these assumptions. The hypothesis of the theorem may be restated in terms of the quotient space $M/G$: we have $\mathcal{H}^{m-2}(M^0) = \mathcal{H}^{m-2}(M^{m-3}) = 0$ if, and only if, $\mathcal{H}^{m-2}(Q^0) = \mathcal{H}^1(Q^{m-3}) = 0$.

In order to make a connection to the Euler-Lagrange equation, we will have to impose further conditions on $M$ and on the group action, but above all, we need a sufficiently weak form of (1). One weak form of the equation is derived as usual by an integration by parts, and it has been computed by Wang [25, 26]. We say that $u \in W^{2,2}(M;N)$ is weakly biharmonic if it satisfies this equation. The resulting theory is not suitable for our purpose, however. We will use another version of the equation derived by the authors [9].

Consider $u \in W^{1,2}(M;N)$. Note that for $\Xi \in L^2(u^{-1}TN)$, the covariant derivative $D\Xi$ is well-defined as a distribution through (3). We write $W^{1,2}(u^{-1}TN)$ for the space of all $\Xi \in L^2(u^{-1}TN)$ such that $D\Xi$ is represented by a section of $T^*M \otimes u^{-1}TN$ with $|D\Xi| \in L^2(M)$. If $\Xi \in W^{1,2}(u^{-1}TN)$, then we can define

$$\Delta\Xi = \operatorname{tr} D^2\Xi$$

similarly, at least as a distribution. If in addition $\Xi \in L^\infty(u^{-1}TN)$, then we can also define

$$J(\Xi) = \Delta\Xi + \operatorname{tr} R(\Xi, du)du.$$

**Definition 3.** *Let $u \in W^{1,2}(M;N)$. An* almost Jacobi field *along $u$ is a section $\Xi \in W^{1,2}(u^{-1}TN) \cap L^\infty(u^{-1}TN)$ such that $J(\Xi) \in L^2(u^{-1}TN)$.*

For $u \in W^{2,2}(M;N)$, the authors have shown [9] that $u$ is weakly biharmonic if, and only if,

$$\Delta \langle \tau(u), \Xi \rangle + 2d^* \langle \tau(u), D\Xi \rangle + \langle \tau(u), J(\Xi) \rangle = 0 \tag{5}$$

for all almost Jacobi fields $\Xi$ along $u$, where $d^*$ is the $L^2$-adjoint of the exterior derivative $d$. But this equation is meaningful for $u \in W^{\tau,2}(M;N)$ as well, and therefore, we can use it to generalise the notion of biharmonic maps.

**Definition 4.** *A map $u \in W^{\tau,2}(M;N)$ is* very weakly biharmonic *if (5) is satisfied for every almost Jacobi field $\Xi$ along $u$.*

**Theorem 5.** *Suppose that $M$ is dilatable and furthermore, that $M^0$ is finite and $M^{m-3} = \emptyset$. Let $u_0 \in C^\infty(M;N)$ be an equivariant map. Then there exists an equivariant, very weakly biharmonic map $u \in K_{u_0}(M;N)$.*

This theorem is proved by showing that under these additional assumptions, the energy minimiser from Theorem 2 solves the Euler-Lagrange equation in the sense of very weakly biharmonic maps. We will see that in the case $m = 3$ (and $Q^0 = \emptyset$), the map obtained in the proof of the theorem is in fact in $W^{2,2}(M; N)$. It then follows that it is a weak solution of the version of the Euler-Lagrange equation derived by Wang [25, 26]. The same is true in the case $m = 4$ if the condition $Q^0 = \emptyset$ is imposed in addition to $Q^1 = \emptyset$. If we only assume that $Q^0$ is finite and $Q^1 = \emptyset$ as in the theorem, then we cannot conclude that we have a map in $W^{2,2}(M; N)$ any more, but we can still prove that we have a very weakly biharmonic map.

Equivariant biharmonic maps have also been studied by Montaldo and Ratto [13] and by Montaldo, Oniciuc, and Ratto [12], although from a different point of view. Furthermore, a different type of equivariant biharmonic maps (called extrinsic) has been studied by Zorn [27] and Cooper [5].

## 2 Preliminaries

In this section, we first recall some of the basic concepts and facts from the theory of transformation groups. We then establish a few further facts that are useful in the context of equivariant biharmonic maps. Finally, we also discuss some tools from geometric measure theory and a curvature functional that we will use later.

Recall that $G$ is a compact Lie group acting by isometries on $M$ and on $N$. Thus for any $a \in G$, there exist two isometries $\lambda_a : M \to M$ and $L_a : N \to N$ representing these group actions. As long as no confusion is likely to arise, we simply write $ax = \lambda_a(x)$ for $x \in M$ and $ay = L_a(y)$ for $y \in N$.

As already mentioned, for any $x \in M$, the orbit $Gx = \{ax \colon a \in G\}$ is an embedded submanifold of $M$. Furthermore, it is a consequence of the Tubular Neighbourhood Theorem [4, Theorem VI.2.2] that an entire tubular neighbourhood of $Gx$ will have orbits of at least the same dimension. Thus the function $x \mapsto \dim Gx$ is lower semicontinuous. If we define $M^j$ as in the introduction, then $M^j \cup \ldots \cup M^m$ is an open set for every $j = \{0, \ldots, m\}$.

Given $a \in G$, we can pull back tangent vector fields on $M$ and on $N$ with the isometries $\lambda_a$ and $L_a$, respectively. If $X$ is a tangent vector field on $M$, then we write

$$a^* X = \lambda_a^* X$$

for $a \in G$ (i.e., we have $a^* X(x) = d\lambda_{a^{-1}}(ax)X(ax)$ for every $x \in M$) and use similar notation for $N$. Given an equivariant map $u : M \to N$, we can also pull back a section $\Xi$ of $u^{-1}TN$, obtaining $a^*\Xi$ with

$$a^*\Xi(x) = dL_{a^{-1}}(u(ax))\Xi(ax), \quad x \in M. \tag{6}$$

If we have an equivariant map $u \in W^{1,2}(M; N)$ and $\Xi \in L^2(u^{-1}TN)$, then we can define $D\Xi$ as a distribution by virtue of (3). There exists a tubular neighbourhood $\mathcal{U}$ of $N$ such that there is a unique, smooth nearest point projection $\pi_N : \mathcal{U} \to N$. We then note that $d\pi_N(y) : T_y\mathcal{U} \to T_yN$ is the orthogonal projection for any $y \in N$. We extend $L_a$ to $\mathcal{U}$, setting $\tilde{L}_a = L_a \circ \pi_N$. Given a map $\Xi : M \to \mathbb{R}^{\bar{n}}$, interpreted as a section of $u^{-1}T\mathbb{R}^{\bar{n}}$, we define $\Xi^\top = d\pi_N(u)\Xi$.

Furthermore, we define $a^*\Xi = a^*\Xi^\top$. With these definitions, a formula similar to (6), but with $L_{a^{-1}}$ replaced by $\tilde{L}_{a^{-1}}$, is true for $\Xi : M \to \mathbb{R}^{\bar{n}}$.

If $\Xi \in W^{1,2}(u^{-1}TN)$ and $X$ is a smooth tangent vector field on $M$, then clearly $\langle D_X\Xi, Z\rangle = \left\langle D_X\Xi, Z^\top\right\rangle$. Moreover, by integration by parts,

$$\int_M \langle D_X\Xi, Z\rangle \, d\mathrm{vol} = -\int_M \left\langle \Xi, D_X Z^\top\right\rangle d\mathrm{vol} - \int_M \left\langle \Xi, Z^\top\right\rangle \operatorname{div} X \, d\mathrm{vol} \qquad (7)$$

for all $Z \in C_0^\infty(\operatorname{Int} M; \mathbb{R}^{\bar{n}})$. If we merely have $\Xi \in L^2(u^{-1}TN)$, then we differentiate $Z^\top = d\pi_N(u)Z$ and use certain observations about the Hessian of the nearest point projection [24, Theorem 2.12.1] to conclude that $\left\langle \Xi, DZ^\top\right\rangle = \langle \Xi, dZ\rangle - \langle A(\Xi, du), Z\rangle$. Hence a distributional version of (7) is still true in this case. In particular $(D_X\Xi)(Z) = (D_X\Xi)(Z^\top)$.

Now if $\Xi$ is in the dual space of $W_0^{1,2}(M;\mathbb{R}^{\bar{n}}) \cap L^\infty(M;\mathbb{R}^{\bar{n}})$, then we can still define $a^*\Xi$ by the condition that

$$a^*\Xi(Z) = \Xi(a_*Z)$$

for all $Z \in C_0^\infty(\operatorname{Int} M; \mathbb{R}^{\bar{n}})$, where $a_*Z = (a^{-1})^*Z$. This way, we can define $a^*D\Xi$ in the distribution sense for any $\Xi \in L^2(u^{-1}TN)$.

We then have a number of identities involving the pull-back of vector fields, which are verified by direct computation if everything is smooth, but which require more careful arguments if we work with less regularity.

**Lemma 6.** *Suppose that $u \in W^{1,2}(M;N)$ is equivariant. Then the following identities hold true for every $a \in G$.*

1. *$a^*(du(X)) = du(a^*X)$ for every smooth tangent vector field $X$ on $M$.*
2. *$a^*D_X\Xi = D_{a^*X}a^*\Xi$ for every $\Xi \in L^2(u^{-1}TN)$ and every smooth tangent vector field $X$ on $M$.*
3. *$a^*\tau(u) = \tau(u)$.*
4. *$a^*\Delta\Xi = \Delta(a^*\Xi)$ for every $\Xi \in W^{1,2}(u^{-1}TN)$.*
5. *$a^*J(\Xi) = J(a^*\Xi)$ for every $\Xi \in W^{1,2}(u^{-1}TN) \cap L^\infty(u^{-1}TN)$.*

*Proof.* 1. By the equivariance, we have $u = L_{a^{-1}} \circ u \circ \lambda_a$. By the chain rule, this implies

$$du(x) = dL_{a^{-1}}(u(ax))du(ax)d\lambda_a(x)$$

for almost every $x \in M$. It follows that

$$\begin{aligned} du(a^*X)(x) &= du(x)d\lambda_{a^{-1}}(ax)X(ax) \\ &= dL_{a^{-1}}(u(ax))du(ax)X(ax) \\ &= (a^*du(X))(x), \end{aligned}$$

as required.

2. First we assume that $\Xi \in W^{1,2}(u^{-1}TN)$. Note that (3) implies that

$$D_X\Xi = (X(\Xi))^\top,$$

where $X(\Xi)$ stands for the component-wise directional derivative of $\Xi$ in the direction $X$. Thus we have

$$D_{a^*X} a^*\Xi = (a^*X(a^*\Xi))^\top = (a^*X(dL_{a^{-1}}(L_a \circ u)(\Xi \circ \lambda_a)))^\top .$$

Because $L_a$ is an isometry on $N$, its Hessian vanishes. Hence the Hessian of $\tilde{L}_a$ satisfies $d\pi_N(ay) Dd\tilde{L}_a(y)(\tilde{\Xi}, \tilde{Z}) = 0$ for any $y \in N$ and $\tilde{\Xi}, \tilde{Z} \in T_y N$. It follows that

$$\begin{aligned} D_{a^*X} a^*\Xi &= \left(a^*X\left(d\tilde{L}_{a^{-1}}(L_a \circ u)(\Xi \circ \lambda_a)\right)\right)^\top \\ &= \left(d\tilde{L}_{a^{-1}}(L_a \circ u)(a^*X(\Xi \circ \lambda_a))\right)^\top \\ &= dL_{a^{-1}}(L_a \circ u)\,(a^*X(\Xi \circ \lambda_a))^\top . \end{aligned}$$

The chain rule implies

$$a^*X(\Xi \circ \lambda_a) = X(\Xi) \circ \lambda_a.$$

Hence

$$D_{a^*X} a^*\Xi = dL_{a^{-1}}(L_a \circ u)(X(\Xi) \circ \lambda_a)^\top = a^*(D_X \Xi).$$

If we only have $\Xi \in L^2(u^{-1}TN)$, then we choose $Z \in C_0^\infty(\operatorname{Int} M; \mathbb{R}^{\bar{n}})$ and we compute

$$\begin{aligned} (a^* D_X \Xi)(Z) &= -\int_M \left\langle \Xi, D_X(a_* Z^\top)\right\rangle \,\mathrm{dvol} - \int_M \left\langle \Xi, a_* Z^\top\right\rangle \operatorname{div} X \, d\mathrm{vol} \\ &= -\int_M \left\langle a^*\Xi, D_{a^*X} Z^\top\right\rangle \,\mathrm{dvol} - \int_M \left\langle a^*\Xi, Z^\top\right\rangle \operatorname{div}(a^*X)\, d\mathrm{vol} \\ &= (D_{a^*X} a^*\Xi)(Z). \end{aligned}$$

In the second step we have used the preceding computations and the fact that $a^* \operatorname{div} X = \operatorname{div}(a^*X)$. This concludes the proof of this statement.

3. Choose local tangent vector fields $e_1, \ldots, e_m$ on $M$ that form an orthonormal basis at every point of their domain. Then

$$\begin{aligned} a^*\tau(u) &= a^*\left(\sum_{\alpha=1}^m (D_{e_\alpha} du(e_\alpha) - du(D_{e_\alpha} e_\alpha))\right) \\ &= \sum_{i=1}^m (D_{a^*e_\alpha} du(a^*e_\alpha) - du(D_{a^*e_\alpha} a^*e_\alpha)) \\ &= \tau(u) \end{aligned}$$

by statements 1 and 2 and the fact that $a^*e_1, \ldots, a^*e_m$ still form an orthonormal basis at every point.

4. This follows from statement 2 similarly to the preceding statement.

5. Choose $e_1, \ldots, e_m$ as before. Then

$$a^*(R(\Xi, du(e_\alpha))du(e_\alpha)) = R(a^*\Xi, du(a^*e_\alpha))du(a^*e_\alpha), \quad i = 1, \ldots, m,$$

by statement 1 and the fact that the curvature is preserved under an isometry. Now we combine this with statement 4, and we obtain the desired formula. □

We now consider the normalised Haar measure (i.e., such that $G$ has measure 1) on $G$. We denote integrals with respect to this measure by

$$\int_G f(a)\,da,$$

where $f : G \to \mathbb{R}^{\bar{n}}$ is an integrable function. For a map

$$\Xi : G \to (W_0^{1,2}(u^{-1}TN) \cap L^\infty(u^{-1}TN))^*,$$

we define $\int_G \Xi(a)\,da$ to be the element of $(W_0^{1,2}(u^{-1}TN) \cap L^\infty(u^{-1}TN))^*$ defined by

$$\left(\int_G \Xi(a)\,da\right)(Z) = \int_G (\Xi(a))(Z)\,da$$

for all $Z \in W_0^{1,2}(u^{-1}TN) \cap L^\infty(u^{-1}TN)$, provided that the integral on the right-hand side exists and does indeed give rise to an element of this space. We then have the following.

**Lemma 7.** *Suppose that $u \in W^{1,2}(M;N)$ is an equivariant map and $\Xi \in W^{1,2}(u^{-1}TN) \cap L^\infty(u^{-1}TN)$. Then*

$$J\left(\int_G a^*\Xi\,da\right) = \int_G a^*J(\Xi)\,da.$$

*Proof.* Let $Z \in W_0^{1,2}(u^{-1}TN) \cap L^\infty(u^{-1}TN)$. Let

$$\bar{\Xi} = \int_G a^*\Xi\,da.$$

Then

$$\begin{aligned} J\left(\bar{\Xi}\right)(Z) &= \operatorname{tr}\int_M \left(\left\langle R\left(\bar{\Xi}, du\right)du, Z\right\rangle - \left\langle D\bar{\Xi}, DZ\right\rangle\right)\,d\mathrm{vol} \\ &= \int_G \operatorname{tr}\int_M \left(\langle R(a^*\Xi, du)du, Z\rangle - \langle D(a^*\Xi), DZ\rangle\right)\,d\mathrm{vol}\,da \\ &= \left(\int_G J(a^*\Xi)\,da\right)(Z) \end{aligned}$$

by the linearity of the integral and Fubini's theorem. Thus

$$J\left(\bar{\Xi}\right) = \int_G J(a^*\Xi)\,da.$$

The claim now follows from Lemma 6. □

When we minimise the functional $E_2$ with the direct method, we may encounter a concentration of $|du|^2$ on a subset of $M$. In order to analyse this concentration set, we need some tools from geometric measure theory. This includes the concept of varifolds, but since we do not need the full theory of varifolds, we give an unconventional definition here. Readers familiar with the theory will nevertheless recognise the concepts. Further information (and the conventional definition) may be found, e.g., in a book by Simon [22].

Let $j \in \{0, \ldots, m\}$. Recall that $\mathcal{H}^j$ denotes the $j$-dimensional Hausdorff measure. A set $\Sigma \subset M$ is called countably $j$-rectifiable if up to an $\mathcal{H}^j$-null set, it is contained in the union of countably many embedded $C^1$-submanifolds of $M$. A rectifiable $j$-varifold in $M$ is a locally $\mathcal{H}^j$-integrable function $\theta : M \to [0, \infty)$ such that $\theta^{-1}((0, \infty))$ is countably $j$-rectifiable.

**Definition 8.** *We say that a varifold $\theta : M \to [0, \infty)$ is* protuberant *if there exists a number $\epsilon > 0$ such that $\theta(x) \in \{0\} \cup [\epsilon, \infty)$ for $\mathcal{H}^j$-almost every $x \in M$.*

If a varifold is protuberant and upper semicontinuous, then $\theta^{-1}((0, \infty))$ is closed. (Readers who prefer to think of varifolds in the conventional terms should define a varifold to be upper semicontinuous if the density function is.)

With any $j$-rectifiable varifold $\theta$ we can associate a Radon measure $\mathcal{H}^j \llcorner \theta$. We can identify Radon measures on $M$ with elements of the dual space $(C_0^0(M))^*$ of $C_0^0(M)$. (Of course $C_0^0(M) = C^0(M)$ as $M$ is compact, but we will replace $M$ by a non-compact manifold later.) For example, the Radon measure $\mathcal{H}^j \llcorner \theta$ corresponds to the functional

$$\eta \mapsto \int_M \eta\theta \, d\mathcal{H}^j, \quad \eta \in C_0^0(M).$$

The space $(C_0^0(M))^*$ is equipped with the weak* topology, and when we speak of convergence of Radon measures, this is the topology that we use.

If $\theta$ is upper semicontinuous and protuberant, then the sets $\theta^{-1}((0, \infty))$ and $\operatorname{supp}(\mathcal{H}^j \llcorner \theta)$ differ by an $\mathcal{H}^j$-null set, so for most purposes, we can exchange one for the other.

It is a well-known fact that a rectifiable $j$-varifold has approximate tangent spaces $\mathcal{H}^j$-almost everywhere on $\Sigma = \theta^{-1}((0, \infty))$. That is, for $\mathcal{H}^j$-almost every $x_0 \in \Sigma$, there exists a $j$-dimensional linear subspace $T_{x_0}\Sigma \subset T_{x_0}M$ such that for all $\eta \in C_0^0(T_{x_0}(M))$,

$$\lim_{r \searrow 0} \int_M r^{-j} \eta \left(r^{-1} \exp_{x_0}^{-1}(x)\right) \theta \, d\mathcal{H}^j(x) = \theta(x_0) \int_{T_{x_0}\Sigma} \eta \, d\mathcal{H}^j.$$

The approximate normal space $T_{x_0}^\perp \Sigma$ is then defined as the orthogonal complement of $T_{x_0}\Sigma$ in $T_{x_0}M$.

Finally, we introduce a curvature functional that we will encounter later. Suppose that $1 \leq j \leq m-1$ and that $\Sigma \subset M$ is a closed, embedded, $j$-dimensional $C^2$-submanifold without boundary. Then it has a mean curvature vector $H$, which may be defined as the trace of the second fundamental form or through the first variation formula

$$\int_\Sigma \operatorname{div}_\Sigma \Phi \, d\mathcal{H}^j = -\int_\Sigma g(\Phi, H) \, d\mathcal{H}^j,$$

which holds for any Lipschitz tangent vector field $\Phi$ on $M$, as this formula characterises $H$. Here, $\operatorname{div}_\Sigma$ denotes the divergence with respect to $\Sigma$. The quantity

$$W(\Sigma) = \frac{1}{2} \int_\Sigma |H|^2 \, d\mathcal{H}^j$$

will play an important role in our analysis. When $j = 1$, it is called the Euler elastica functional and when $j = 2$, it is called the Willmore functional.

# 3 Minimisation of a relaxed functional

Finding minimisers of the the functional $E_2$ is not easy, because the functional is not coercive on the usual Sobolev spaces. For example, a bound on $E_2(u)$ will not entail a bound on $\|u\|_{W^{2,2}(M)}$. Furthermore, there are obvious minimizers of $E_2$, namely solutions of the equation $\tau(u) = 0$, that do not belong to $W^{2,2}(M;N)$. (This is a consequence of an example constructed by Rivière [20].) These observations suggest that we should work with the space $W^{\tau,2}(M;N)$ instead. But this space is difficult to work with, as the condition $\tau(u) \in L^2(u^{-1}TN)$ gives no additional regularity for the lower order derivatives (not even if $\tau(u) = 0$, as shown by Rivière's work [20] again). The situation is somewhat better in this respect if we work in the space $K_{u_0}(M;N)$ for a suitable map $u_0 : M \to N$. This, however, gives rise to other difficulties. In particular, we have a lack of compactness due to energy concentration here. In order to overcome this problem, we use tools from geometric measure theory, encoding the concentrated energy in a measure on $M$. We follow an approach going back to Lin [11] and to Ambrosio and Soner [1] and developed further for the problem of biharmonic maps by the authors [14, 8].

In order to avoid the need to treat $\partial M$ separately, we extend $M$ across the boundary. That is, we choose an $m$-dimensional Riemannian manifold $M'$ without boundary such that $M$ is a compact subset of $M'$. Suppose that $u_0 \in C^\infty(M;N)$ and choose a smooth extension of $u_0$ to $M'$ (also denoted by $u_0$ for simplicity). If we have a map $u \in K_{u_0}(M;N)$, then its extension to $M'$ with $u = u_0$ in $M' \backslash M$ will belong to $K(M'';N)$ for any open set $M'' \subset M'$ such that $\overline{M''}$ is compact. Using this observation, we can mostly work with $M'$ rather than $M$ and ignore the boundary in this section.

For a vector bundle $\pi : F \to M'$ over $M'$ and $k \in \mathbb{N}_0$, let $\Gamma_0^k(F)$ denote the space of all $k$ times continuously differentiable sections of $F$ with compact support. Consider the dual space $(\Gamma_0^0(TM' \otimes TM'))^*$ of $\Gamma_0^0(TM' \otimes TM')$ and the subspace $\mathcal{M}(M')$ comprising all $\mu \in (\Gamma_0^0(TM' \otimes TM'))^*$ that are symmetric and positive semidefinite in the sense that

$$\mu(X_1 \otimes X_2) = \mu(X_2 \otimes X_1) \quad \text{and} \quad \mu(X \otimes X) \geq 0$$

for all $X, X_1, X_2 \in \Gamma_0^0(TM')$. Then for any $\mu \in \mathcal{M}(M')$, we can find a Radon measure $\bar{\mu}$ on $M'$ and a $\bar{\mu}$-measurable section $\sigma$ of $T^*M' \otimes T^*M'$ such that $\operatorname{tr} \sigma = 1$ almost everywhere and

$$\mu(X_1 \otimes X_2) = \int_{M'} \sigma(X_1, X_2)\, d\bar{\mu}$$

for all $X_1, X_2 \in \Gamma_0^0(TM')$. We then write $\mu = \bar{\mu} \llcorner \sigma$.

To any $u \in W^{1,2}(M';N)$, we can associate an element $\hat{\mu}_u \in \mathcal{M}(M')$ with

$$\hat{\mu}_u(X_1 \otimes X_2) = \int_{M'} \langle du(X_1), du(X_2) \rangle \, d\mathrm{vol}$$

for $X_1, X_2 \in \Gamma_0^0(TM')$. Furthermore, any rectifiable $(m-2)$-varifold $\theta$ will give rise to an element $\check{\mu}_\theta \in \mathcal{M}(M')$ as follows. Let $\Sigma = \theta^{-1}((0,\infty))$. For $x \in \Sigma$, let $\pi_x^\perp$ denote the orthogonal projection onto the approximate normal space $T_x^\perp \Sigma$ (which is well-defined $\mathcal{H}^{m-2}$-almost everywhere). If $\sigma$ is the section of

$T^*M' \otimes T^*M'$ with $\sigma(x)(X_1, X_2) = g(\pi_x^\perp X_1, \pi_x^\perp X_2)$ for almost all $x \in \Sigma$ and all $X_1, X_2 \in T_xM'$, then $\check{\mu}_\theta = \mathcal{H}^{m-2} \llcorner \theta\sigma$ belongs to $\mathcal{M}(M')$.

We define a map $\delta : \mathcal{M}(M') \to (\Gamma_0^1(TM'))^*$ as follows: if $\mu = \bar{\mu} \llcorner \sigma$ for a Radon measure $\bar{\mu}$ on $M'$ and a $\bar{\mu}$-measurable section $\sigma$ of $T^*M' \otimes T^*M'$ with $\operatorname{tr} \sigma = 1$ almost everywhere, then

$$\delta\mu(X) = \int_{M'} \left( \operatorname{tr} \sigma(\nabla X) - \frac{1}{2} \operatorname{div} X \right) d\bar{\mu}, \quad X \in \Gamma_0^1(TM').$$

If $\hat{\mu}_u$ corresponds to a map $u \in K(M'; N)$, then we have

$$\delta\hat{\mu}_u(X) = -\int_{M'} \langle du(X), \tau(u) \rangle \, d\mathrm{vol}$$

by (4). If $\check{\mu}_\theta$ corresponds to a rectifiable $(m-2)$-varifold $\theta$, then

$$\delta\check{\mu}_\theta(X) = -\int_\Sigma \theta \operatorname{div}_\Sigma X \, d\mathcal{H}^{m-2},$$

where $\Sigma = \theta^{-1}((0, \infty))$.

Next, we define a functional $\mathcal{W} : \mathcal{M}(M') \to [0, \infty]$ as follows: if $\mu = \bar{\mu} \llcorner \sigma$ as above, then

$$\mathcal{W}(\mu) = \frac{1}{2} \sup \left\{ (\delta\mu(X))^2 \colon X \in \Gamma_0^1(TM') \text{ with } \int_{M'} \sigma(X, X) \, d\bar{\mu} \le 1 \right\}.$$

If $\hat{\mu}_u$ belongs to a map $u \in K(M'; N)$, then [14]

$$\mathcal{W}(\hat{\mu}_u) \le \frac{1}{2} \int_{M'} |\tau(u)|^2 \, d\mathrm{vol}.$$

If we have a rectifiable $(m-2)$-varifold $\theta : M \to \{0, 1\}$ such that $\Sigma = \theta^{-1}(\{1\})$ is a $C^2$-submanifold of $M'$, then $\mathcal{W}(\check{\mu}_\theta) = W(\Sigma)$ for the functional $W$ introduced in Sect. 2.

If $\mathcal{W}(\mu) < \infty$, then we can say something about the structure of $\delta\mu$. The following is a result of the second author [14, Proposition 2.1].

**Lemma 9.** *Suppose that $\mu \in \mathcal{M}(M')$ is of the form $\mu = \bar{\mu} \llcorner \sigma$ for a Radon measure $\bar{\mu}$ on $M'$ and a $\bar{\mu}$-measurable section $\sigma$ of $T^*M' \otimes T^*M'$ with $\operatorname{tr} \sigma = 1$ almost everywhere. For $x \in M'$, let $\mathcal{N}_x$ be the null space of $\sigma(x)$. Then there exists a unique $\bar{\mu}$-measurable section $H$ of $TM'$ such that $H(x) \perp \mathcal{N}_x$ for $\bar{\mu}$-almost every $x \in M'$ and*

$$\delta\mu(X) = \int_{M'} \sigma(X, H) \, d\bar{\mu}$$

*for every $X \in \Gamma_0^1(TM')$. Furthermore,*

$$\mathcal{W}(\mu) = \frac{1}{2} \int_{M'} \sigma(H, H) \, d\bar{\mu}.$$

If $\mu$ belongs to a $C^2$-submanifold of $M'$, then $H$ is the mean curvature vector. In general, we think of it as a generalised mean curvature vector.

For any Borel set $B \subset M'$, we define

$$\mathcal{W}(\mu; B) = \frac{1}{2} \int_B \sigma(H, H) \, d\bar{\mu}.$$

If we have both a map $u \in W^{1,2}(M'; N)$ and a rectifiable $(m-2)$-varifold $\theta$, then we also define

$$W(u, \theta; B) = \mathcal{W}(\hat{\mu}_u + \check{\mu}_\theta; B).$$

If $B$ is an open set such that $\mathcal{W}(\hat{\mu}_u; B) < \infty$ and $\mathcal{W}(\check{\mu}_\theta; B) < \infty$, then it is easy to see that $W(u, \theta; B) = \mathcal{W}(\hat{\mu}_u; B) + \mathcal{W}(\check{\mu}_\theta; B)$. On the other hand, it may happen that $W(u, \theta; B) < \infty$ even if $\mathcal{W}(\hat{\mu}_u; B) = \infty$ and $\mathcal{W}(\check{\mu}_\theta; B) = \infty$.

The following is a consequence of the results of the first author [8]. Although the proofs in that paper are carried out for the special case of a flat domain only, it is not difficult to see that they can be generalised.

**Theorem 10.** *Suppose that $M$ is dilatable and $u_0 \in C^\infty(M; N)$. Let $(u_k)_{k \in \mathbb{N}}$ be a sequence in $K_{u_0}(M; N)$ with*

$$\limsup_{k \to \infty} E_2(u_k) < \infty.$$

*Then there exists a subsequence $(u_{k_\ell})_{\ell \in \mathbb{N}}$ and there exist a map $u \in W^{\tau,2}_{u_0}(M; N)$ and a protuberant, upper semicontinuous, rectifiable $(m-2)$-varifold $\theta$ such that $u_{k_\ell} \rightharpoonup u$ weakly in $W^{1,2}(M; \mathbb{R}^{\bar{n}})$ and $\tau(u_{k_\ell}) \rightharpoonup \tau(u)$ weakly in $L^2(M; \mathbb{R}^{\bar{n}})$, as well as $\hat{\mu}_{u_k} \overset{*}{\rightharpoonup} \hat{\mu}_u + \check{\mu}_\theta$ weakly* in $(\Gamma^0_0(T^*M' \otimes T^*M'))^*$ and*

$$W(u, \theta; M) \leq \liminf_{\ell \to \infty} E_2(u_k).$$

We will apply this theorem to minimising sequences of $E_2$ in suitable spaces. The result can also be formulated in a more general form, in which case it allows to minimise the functional $\mathcal{W}(\cdot\,; M)$ with the direct method from the calculus of variations. While it is difficult to say anything about the regularity of the minimisers in general, if we already have some regularity for $\theta$, then some degree of regularity for $u$ follows as well.

**Lemma 11.** *Suppose that $u \in W^{\tau,2}(M'; N)$. Moreover, suppose that $\Sigma \subset M'$ is a smooth $(m-2)$-dimensional submanifold and $\theta : M \to [0, \infty)$ is a function such that $\theta|_\Sigma$ is continuous and $\theta|_{M' \setminus \Sigma} = 0$. If $W(u, \theta; M') < \infty$, then there exists a closed set $\Sigma' \subset M'$ with $\mathcal{H}^{m-2}(\Sigma') = 0$ and $u \in W^{2,2}_{\mathrm{loc}}(M' \setminus \Sigma'; N)$.*

*Proof.* Combine the results of the first author [8, Lemma 12 and Remark 4] with the regularity results of the second author [16, 18]. ☐

# 4 The Euler-Lagrange equation

Once we have proved Theorem 2, we will show that under the hypothesis of Theorem 5, the minimisers of $E_2$ are very weakly biharmonic maps. To this end, we need to study the Euler-Lagrange equation in both the weak and very weak form.

**Lemma 12.** *Suppose that $u \in W^{2,2}(M;N)$ is a map such that for every $\eta \in C_0^\infty(M)$ and every smooth tangent vector field $\Upsilon$ on $N$,*

$$\int_M \langle \tau(u), J(\eta\,\Upsilon \circ u)\rangle \, d\mathrm{vol} = 0.$$

*Then $u$ is weakly biharmonic.*

*Proof.* It is readily checked that $u$ is weakly biharmonic if

$$\int_M \langle \tau(u), J(\Xi)\rangle \, d\mathrm{vol} = 0 \tag{8}$$

for any $\Xi \in W_0^{2,2}(u^{-1}TN)$.

Denote by $\bar{e}_i$ the $i$-th standard unit vector in $\mathbb{R}^{\bar{n}}$. Given $\Xi \in W_0^{2,2}(u^{-1}TN)$, there exist unique functions $\Xi^i \in W_0^{2,2}(M)$ such that

$$\Xi = \sum_{i=1}^{\bar{n}} \Xi^i \bar{e}_i.$$

For every $i = 1, \ldots, \bar{n}$, choose a sequence of functions $\xi_k^i \in C_0^\infty(M)$ such that $\xi_k^i \to \Xi^i$ in $W^{2,2}(M)$. For $y \in N$, define $\Upsilon_i(y) = d\pi_N(y)\bar{e}_i$. Under the hypothesis of the lemma, we have

$$\int_M \big\langle \tau(u), J(\xi_k^i\, \Upsilon_i \circ u)\big\rangle \, d\mathrm{vol} = 0$$

for every $k \in \mathbb{N}$ and $i = 1, \ldots, \bar{n}$. Letting $k \to \infty$, we obtain (8). □

**Lemma 13.** *A map $u \in W^{\tau,2}(M;N)$ is very weakly biharmonic if, and only if, for every almost Jacobi field $\Xi \in W^{1,2}(u^{-1}TN)$ that vanishes in a neighbourhood of $\partial M$, the equation*

$$\int_M \langle \tau(u), J(\Xi)\rangle \, d\mathrm{vol} = 0 \tag{9}$$

*holds true.*

*Proof.* If $u$ is very weakly biharmonic and $\Xi$ is an almost Jacobi field that vanishes in a neighbourhood of $\partial M$, then $\langle \tau(u), J(\Xi)\rangle$ is a divergence term by (5) and so (9) follows.

Conversely, suppose that (9) holds for any almost Jacobi field that vanishes in a neighbourhood of the boundary. Let $\Xi \in W^{1,2}(u^{-1}TN) \cap L^\infty(u^{-1}TN)$ be an almost Jacobi field. Note that for any $\eta \in C_0^\infty(\operatorname{Int} M)$, the vector field $\eta\Xi$ is automatically almost Jacobi as well. Equation (9), applied to $\eta\Xi$, then amounts to the very weak form of the Euler-Lagrange equation, since $\eta$ can be chosen arbitrarily. □

We now show that for an equivariant map that minimises $E_2$ among all other equivariant maps, the Euler-Lagrange equation holds, provided that we have enough regularity.

**Proposition 14.** *Suppose that $u \in W^{2,2}(M;N)$ is an equivariant map such that $E_2(u) \le E_2(v)$ for any other equivariant map $v \in W^{2,2}(M;N)$ that agrees with $u$ in a neighbourhood of $\partial M$. Then $u$ is weakly biharmonic.*

*Proof.* Let $\eta \in C_0^\infty(\operatorname{Int} M)$ and let $\Upsilon$ be a smooth tangent vector field on $N$. Define $F(x,y) = \eta(x)\Upsilon(y)$ for $x \in M$ and $y \in N$. Furthermore, define

$$\bar{F}(x,y) = \int_G dL_{a^{-1}}(ay)F(ax,ay)\,da.$$

Then for any $a \in G$, we have

$$\begin{aligned} dL_{a^{-1}}(ay)\bar{F}(ax,ay) &= \int_G dL_{a^{-1}}(ay)dL_{b^{-1}}(bay)F(bax,bay)\,db \\ &= \int_G dL_{a^{-1}}(ay)dL_{ab^{-1}}(by)F(bx,by)\,db \\ &= \int_G dL_{b^{-1}}(by)F(bx,by)\,db \\ &= \bar{F}(x,y) \end{aligned} \tag{10}$$

by the chain rule.

Let $\bar{\Phi}_t : M \times N \to N$ be the map such that $\bar{\Phi}_t(x,\cdot)$ is the flow on $N$ generated by $\bar{F}(x,\cdot)$ for all $x \in M$. We claim that

$$\bar{\Phi}_t(ax,ay) = a\bar{\Phi}_t(x,y) \tag{11}$$

for all $a \in G$, $x \in M$, $y \in N$, and $t \in \mathbb{R}$. In order to prove this, let $\bar{\Psi}_t(x,y) = a^{-1}\bar{\Phi}_t(ax,ay)$. Then $\bar{\Psi}_0(x,y) = y$ and

$$\begin{aligned} \frac{\partial}{\partial t}\bar{\Psi}_t(x,y) &= \frac{\partial}{\partial t}L_{a^{-1}}(\bar{\Phi}_t(ax,ay)) \\ &= dL_{a^{-1}}(a\bar{\Psi}_t(x,y))\bar{F}(ax,a\bar{\Psi}_t(x,y)) \\ &= \bar{F}(x,\bar{\Psi}_t(x,y)) \end{aligned}$$

by (10). Hence $\bar{\Psi}_t(x,\cdot)$ is the unique flow generated by $\bar{F}(x,\cdot)$, which implies (11).

Let $\bar{u}_t(x) = \bar{\Phi}_t(x,u(x))$ for $x \in M$. Then for any $t$, we find that $\bar{u}_t \in W^{2,2}(M;N)$ is an equivariant map. Hence

$$\left.\frac{d}{dt}\right|_{t=0} E_2(\bar{u}_t) = 0$$

if the derivative exists. We claim that it does exist and that

$$\left.\frac{d}{dt}\right|_{t=0} E_2(\bar{u}_t) = \int_M \langle \tau(u), J(F \circ (\mathrm{id}_M \times u))\rangle\,d\mathrm{vol}. \tag{12}$$

In order to verify this formula, we first assume that $u$ is smooth. Then by computations of Jiang [10], we have

$$\frac{\partial}{\partial t}|\tau(\bar{u}_t)|^2 = 2\left\langle \tau(\bar{u}_t), J(\bar{F} \circ (\mathrm{id}_M \times \bar{u}_t))\right\rangle.$$

Furthermore, it is readily checked that

$$\left\langle \tau(\bar{u}_t), J(\bar{F} \circ (\mathrm{id}_M \times \bar{u}_t))\right\rangle \le C\left(|\nabla du|^2 + |du|^4 + 1\right) \tag{13}$$

for all $t$ in an interval $(-c, c)$ for two constants $C, c > 0$ that are independent of $u$.

Now for $u \in W^{2,2}(M; N)$, we can find a sequence of smooth maps $u_k \in C^\infty(M; N)$ converging to $u$ in the strong $W^{2,2}$-topology. This is possible because $m \leq 4$ and the sequence can be constructed with a method of Schoen and Uhlenbeck [21, Sect. 4] (adapted to $W^{2,2}$-maps). Then $\bar{\Phi}_t \circ (\mathrm{id}_M \times u_k) \to \bar{\Phi}_t \circ (\mathrm{id}_M \times u)$ in $W^{2,2}(M; \mathbb{R}^\ell)$ as well for every $t \in \mathbb{R}$. Using (13) and the dominated convergence theorem, we now obtain

$$E_2(\bar{u}_T) - E_2(u) = \int_0^T \int_M \left\langle \tau(\bar{u}_t), J(\bar{F} \circ (\mathrm{id}_M \times \bar{u}_t)) \right\rangle \, d\mathrm{vol}\, dt$$

whenever $|T|$ is sufficiently small. It follows that

$$\left. \frac{d}{dt} \right|_{t=0} E_2(\bar{u}_t) = \int_M \left\langle \tau(u), J(\bar{F} \circ (\mathrm{id}_M \times u)) \right\rangle \, d\mathrm{vol}.$$

By the equivariance of $u$, we have

$$\bar{F} \circ (\mathrm{id}_M \times u) = \int_G a^*(F \circ (\mathrm{id}_M \times u)) \, da.$$

Hence by Lemma 7,

$$J(\bar{F} \circ (\mathrm{id}_M \times u)) = \int_G a^* J(F \circ (\mathrm{id}_M \times u)) \, da.$$

By Lemma 6, we have $\tau(u) = a^*\tau(u)$ for every $a \in G$. Hence

$$\begin{aligned} \left\langle \tau(u), J(\bar{F} \circ (\mathrm{id}_M \times u)) \right\rangle &= \int_G \langle a^*\tau(u), a^* J(F \circ (\mathrm{id}_M \times u)) \rangle \, da \\ &= \langle \tau(u), J(F \circ (\mathrm{id}_M \times u)) \rangle , \end{aligned}$$

and we obtain (12). Lemma 12 then implies that $u$ is weakly biharmonic. $\square$

Since the hypothesis of Proposition 14 will not necessarily be satisfied when we use the result in the proof of Theorem 5, we also need the following statement.

**Lemma 15.** *Let $m = 4$ and suppose that $S \subset \operatorname{Int} M$ is a finite set. If $u \in W^{\tau,2}(M; N) \cap W^{2,2}_{\mathrm{loc}}(M \backslash S; N)$ is very weakly biharmonic in $M \backslash S$, then it is very weakly biharmonic in $M$.*

*Proof.* For $r > 0$, let $B_r(S) = \bigcup_{x \in S} B_r(x)$ denote the union of the balls of radius $r$ about the points of $S$.

Let $\Xi \in W^{1,2}(u^{-1}TN) \cap L^\infty(u^{-1}TN)$ be an almost Jacobi field and $\eta \in C_0^\infty(\operatorname{Int} M)$. Fix $r > 0$ and choose a cut-off function $\chi \in C_0^\infty(\operatorname{Int} M)$ with $\chi \equiv 1$ in $M \backslash B_r(S)$ and $\chi \equiv 0$ in $B_{r/2}(S)$. This function may be chosen such that

$$|d\chi|^2 + |\nabla d\chi| \leq \frac{C_1}{r^2}$$

for a constant $C_1$ that depends only on $M$ and $S$.

Let $h$ denote the bundle metric on $T^*M$ induced by $g$. According to (5), we have

$$\int_M \left(\Delta(\chi\eta)\,\langle\tau(u),\Xi\rangle + 2h(d(\chi\eta),\langle\tau(u),D\Xi\rangle) + \chi\eta\,\langle\tau(u),J(\Xi)\rangle\right)\,d\mathrm{vol} = 0.$$

Thus

$$\begin{aligned}\int_M \chi\left(\Delta\eta\,\langle\tau(u),\Xi\rangle + 2h(d\eta,\langle\tau(u),D\Xi\rangle) + \eta\,\langle\tau(u),J(\Xi)\rangle\right)\,d\mathrm{vol}\\ = -\int_M \left((\eta\Delta\chi + 2h(d\eta,d\chi))\,\langle\tau(u),\Xi\rangle + 2\eta h(d\chi,\langle\tau(u),D\Xi\rangle)\right)\,d\mathrm{vol}.\end{aligned}$$

We have

$$\begin{aligned}&\left|\int_M \eta\Delta\chi\,\langle\tau(u),\Xi\rangle\;d\mathrm{vol}\right|\\ &\qquad\le \|\eta\|_{L^\infty(M)}\|\Xi\|_{L^\infty(M)}\left(\int_{B_r(S)}|\Delta\chi|^2\,d\mathrm{vol}\right)^{\frac{1}{2}}\left(\int_{B_r(S)}|\tau(u)|^2\,d\mathrm{vol}\right)^{\frac{1}{2}}.\end{aligned}$$

It follows that there exists a constant $C_2$ that depends only on $M$, $S$, $\Xi$, and $\eta$, such that

$$\left|\int_M \eta\Delta\chi\,\langle\tau(u),\Xi\rangle\;d\mathrm{vol}\right| \le C_2\left(\int_{B_r(S)}|\tau(u)|^2\,d\mathrm{vol}\right)^{\frac{1}{2}}.$$

Similarly, we find that

$$\left|\int_M h(d\eta,d\chi)\,\langle\tau(u),\Xi\rangle\;d\mathrm{vol}\right| \le C_3 r\left(\int_{B_r(S)}|\tau(u)|^2\,d\mathrm{vol}\right)^{\frac{1}{2}}$$

and

$$\begin{aligned}&\left|\int_M \eta h(d\chi,\langle\tau(u),D\Xi\rangle)\,d\mathrm{vol}\right|\\ &\qquad\qquad\le C_4\left(\int_M |d\chi|^2|D\Xi|^2\,d\mathrm{vol}\right)^{\frac{1}{2}}\left(\int_{B_r(S)}|\tau(u)|^2\,d\mathrm{vol}\right)^{\frac{1}{2}}\end{aligned}$$

for other constants $C_3 = C_3(M,S,\Xi,\eta)$ and $C_4 = C_4(\eta)$. Since we have $u \in W^{2,2}_{\mathrm{loc}}(M\backslash S;N)$, it follows that $\Xi \in W^{2,2}_{\mathrm{loc}}(M\backslash S;\mathbb{R}^{\bar{n}})$ [9, Sect. 4]. Therefore, we can integrate by parts to obtain

$$\begin{aligned}\int_M |d\chi|^2|D\Xi|^2\,d\mathrm{vol} = \int_M |d\chi|^2\,\langle \mathrm{tr}\,R(\Xi,du)du - J(\Xi),\Xi\rangle\;d\mathrm{vol}\\ - \int_M h(d|d\chi|^2,\langle\Xi,D\Xi\rangle)\,d\mathrm{vol}.\end{aligned}$$

We then see that

$$\int_M |d\chi|^2|D\Xi|^2\,d\mathrm{vol} \le C_4\left(\int_M |d\chi|^2|du|^2\,d\mathrm{vol} + 1\right)$$

for a constant $C_4 = C_4(M, S, \Xi)$. Using a simple generalisation of the monotonicity formula derived by Große-Brauckmann [7] (see also a version derived by the second author [15, Lemma 4.1]), we find that

$$\int_M |d\chi|^2 |du|^2 \, d\mathrm{vol} \le \frac{C_5}{r^2} \int_{B_r(S)} |du|^2 \, d\mathrm{vol} \le C_6(E_1(u) + E_2(u))$$

for two constants $C_5$ and $C_6$ depending only on $M$ and $S$. We eventually obtain a constant $C_7$, depending only on $M$, $N$, $S$, $u$, $\Xi$, and $\eta$ (but not on $r$), such that

$$\left| \int_M \chi \left( \Delta\eta \langle \tau(u), \Xi \rangle + 2h(d\eta, \langle \tau(u), D\Xi \rangle) + \eta \langle \tau(u), J(\Xi) \rangle \right) d\mathrm{vol} \right| \\ \le C_7 \left( \int_{B_r(S)} |\tau(u)|^2 \, d\mathrm{vol} \right)^{\frac{1}{2}}.$$

When we let $r$ tend to 0, then the right-hand side will converge to 0. Hence the inequality implies that

$$\int_M \left( \Delta\eta \langle \tau(u), \Xi \rangle + 2h(d\eta, \langle \tau(u), D\Xi \rangle) + \eta \langle \tau(u), J(\Xi) \rangle \right) d\mathrm{vol} = 0,$$

which amounts to (5). So $u$ is a very weakly biharmonic map. $\square$

# 5 An estimate for the elastica/Willmore functional

When using Theorem 10 in order to prove Theorem 2 and Theorem 5, we will find that the countably rectifiable set $\Sigma = \theta^{-1}((0, \infty))$ may include some $(m-2)$-dimensional orbits of the group action on $M$. Since we know that these orbits are smooth submanifolds, each of them can be understood individually with the help of Lemma 11. But we also need to show that they do not collectively pose a problem, and to this end, we need to estimate the Euler elastica functional for curves and the Willmore functional for surfaces.

The following inequality follows from a variant of a formula due to Simon [23]. We will use it for curves in the case $m = 3$ and for surfaces in the case $m = 4$, although the result is valid for ambient manifolds of any dimension.

**Lemma 16.** *Let $j = 1$ or $j = 2$. There exists a constant $C$ with the following property. Suppose that $x_0 \in M$ and let $R > 0$ be the injectivity radius of $x_0$. Suppose that $r \in (0, \frac{R}{2}]$. Let $\Sigma \subset M$ be an embedded $j$-dimensional $C^2$-submanifold that is closed relative to $B_r(x_0)$ and with $x_0 \in \Sigma$. Let $H$ denote the mean curvature vector of $\Sigma$. Let $\omega = 2$ if $j = 1$ and $\omega = \pi$ if $j = 2$. Then*

$$r^j \omega \le (2 + Cr^2)\mathcal{H}^j(\Sigma \cap B_r(x_0)) + \frac{r^2}{2j^2} \int_{\Sigma \cap B_r(x_0)} |H|^2 \, d\mathcal{H}^j.$$

*Proof.* It suffices to check that Simon's arguments [23, Section 1] carry over to the case of a non-flat manifold with minor modifications. For the convenience of the reader, we carry out the proof nevertheless.

The arguments are based on the first variation identity

$$\int_\Sigma \operatorname{div}_\Sigma \Phi \, d\mathcal{H}^2 = -\int_\Sigma g(\Phi, H) \, d\mathcal{H}^2, \tag{14}$$

which is valid for any Lipschitz tangent vector field $\Phi$ on $M$ with support in $\overline{B_r(x_0)}$. Choose normal coordinates $x$ about $x_0$ and observe that in these coordinates, the metric tensor satisfies

$$g_{\alpha\beta} = \delta_{\alpha\beta} + h_{\alpha\beta}$$

for a tensor $h$ with $|h_{\alpha\beta}| \leq C_1|x|^2$ and $\left|\frac{\partial h_{\alpha\beta}}{\partial x^\gamma}\right| \leq C_1|x|$ in $B_r(0)$, $\alpha, \beta, \gamma = 1, \ldots, m$, for some constant $C_1$ that depends only on the geometry of $M$.

Let $s \in (0, r)$. Define the function

$$f(\rho) = \begin{cases} \frac{1}{s^j} - \frac{1}{r^j} & \text{if } \rho \leq s, \\ \frac{1}{\rho^j} - \frac{1}{r^j} & \text{if } s < \rho < r, \\ 0 & \text{if } \rho \geq r. \end{cases}$$

Let $X(x) = \sum_{\alpha=1}^m x^\alpha \frac{\partial}{\partial x^\alpha}$ and choose $\Phi = f(|X|)X$ in (14). A direct calculation then shows that

$$\left| \operatorname{div}_\Sigma \Phi - \frac{j}{s^j} + \frac{j}{r^j} \right| \leq \frac{C_2|x|^2}{s^j}$$

on $\Sigma \cap B_s(x_0)$, whereas

$$\left| \operatorname{div}_\Sigma \Phi - \frac{j|X^\perp|^2}{|X|^{j+2}} + \frac{j}{r^j} \right| \leq C_3|x|^{2-j}$$

on $\Sigma \cap B_r(x_0) \backslash \overline{B_s(x_0)}$, where $X^\perp$ stands for the orthogonal projection of $X$ onto the normal space of $\Sigma$ and $C_2, C_3$ are two constants that depend only on $M$. Hence there exists a function $a : \overline{B_r(x_0)} \to \mathbb{R}$ with

$$\sup_{B_r(x_0) \backslash \overline{B_s(x_0)}} |a| \leq C_4 = C_4(M),$$

such that whenever $\partial B_s(x_0)$ and $\partial B_r(x_0)$ intersect $\Sigma$ transversally, it follows that

$$\begin{aligned} \frac{j}{s^j} \mathcal{H}^j(\Sigma \cap B_s(x_0)) &+ j \int_{\Sigma \cap B_r(x_0) \backslash B_s(x_0)} \frac{|X^\perp|^2}{|X|^{j+2}} \, d\mathcal{H}^j \\ &= \frac{j}{r^j} \mathcal{H}^j(\Sigma \cap B_r(x_0)) - \int_{\Sigma \cap B_r(x_0)} \left( jr^{2-j} a + f(|X|) g(X, H) \right) \, d\mathcal{H}^j. \end{aligned}$$

We have the identity

$$\frac{|X^\perp|^2}{|X|^{j+2}} + \frac{g(X, H)}{j|X|^j} = \left| \frac{X^\perp}{|X|^{j/2+1}} + |X|^{1-j/2} \frac{H}{2j} \right|^2 - |X|^{2-j} \frac{|H|^2}{4j^2}.$$

Therefore,

$$\begin{aligned}
&\frac{1}{s^j}\mathcal{H}^j(\Sigma\cap B_s(x_0)) + \int_{\Sigma\cap B_r(x_0)\setminus B_s(x_0)} \left|\frac{X^\perp}{|X|^{j/2+1}} + |X|^{1-j/2}\frac{H}{2j}\right|^2 d\mathcal{H}^j \\
&\qquad = \frac{1}{r^j}\mathcal{H}^j(\Sigma\cap B_r(x_0)) + \int_{\Sigma\cap B_r(x_0)\setminus B_s(x_0)} |X|^{2-j}\frac{|H|^2}{4j^2}\,d\mathcal{H}^j \\
&\qquad\quad + \int_{\Sigma\cap B_r(x_0)} \left(\frac{g(X,H)}{jr^j} - r^{2-j}a\right) d\mathcal{H}^j - \int_{\Sigma\cap B_s(x_0)} \frac{g(X,H)}{js^j}\,d\mathcal{H}^j.
\end{aligned}$$

Now we let $s \to 0$. Since $x_0 \in \Sigma$, this gives

$$\begin{aligned}
&\omega + \int_{\Sigma\cap B_r(x_0)} \left|\frac{X^\perp}{|X|^{j/2+1}} + |X|^{1-j/2}\frac{H}{2j}\right|^2 d\mathcal{H}^j \\
&= \frac{1}{r^j}\mathcal{H}^2(\Sigma\cap B_r(x_0)) + \int_{\Sigma\cap B_r(x_0)} \left(|X|^{2-j}\frac{|H|^2}{4j^2} + \frac{g(X,H)}{jr^j} - r^{2-j}a\right) d\mathcal{H}^j.
\end{aligned}$$

We drop the square term on the left-hand side and use Young's inequality to derive the estimate

$$\frac{g(X,H)}{jr^j} \leq \frac{|X|^2}{r^{2+j}} + \frac{r^{2-j}|H|^2}{4j^2}.$$

Furthermore, we recall the estimate for the supremum of $|a|$. Since $|X| \leq r$ in $B_r(x_0)$, we immediately obtain the desired inequality if $\partial B_r(x_0)$ intersects $\Sigma$ transversally. If it does not, then we first prove the inequality for a sequence of radii $r_k \to r$ and then take the limit. □

# 6 Proof of Theorem 2 and Theorem 5

Suppose that $u_0 \in C^\infty(M;N)$ is an equivariant map. Let $\mathcal{A}_{u_0}$ denote the set of all equivariant maps in $K_{u_0}(M;N)$. We want to minimise the energy $E_2$ in $\mathcal{A}_{u_0}$. Once we have achieved this, we have a proof of Theorem 2. We then show that the minimiser is a very weakly biharmonic map under the hypothesis of Theorem 5.

Choose a minimising sequence $(u_k)_{k\in\mathbb{N}}$. After discarding a subsequence, we have the convergence described in Theorem 10. That is, there exists a map $u \in W^{\tau,2}(M;N)$ such that $u_k \rightharpoonup u$ weakly in $W^{1,2}(M;N)$ and $\tau(u_k) \rightharpoonup \tau(u)$ weakly in $L^2(u^{-1}TN)$. Furthermore, there exists a protuberant, upper semicontinuous, rectifiable $(m-2)$-varifold $\theta$ such that $\hat{\mu}_{u_k} \overset{*}{\rightharpoonup} \hat{\mu}_u + \check{\mu}_\theta$ and

$$W(u,\theta;M) \leq \liminf_{k\to\infty} E_2(u_k).$$

Clearly $u$ is equivariant.

Let $\Sigma = \operatorname{supp}(\mathcal{H}^{m-2} \llcorner \theta)$. Since $\Sigma$ arises through energy concentration of the equivariant maps $u_k$, it is invariant under the action of $G$ and $\theta$ is constant on $Gx$ for each $x \in \Sigma$. In particular, we have $Gx \subset \Sigma$ whenever $x \in \Sigma$. On the other hand, we have $\mathcal{H}^{m-2}(\Sigma) < \infty$. Therefore, we conclude that $\Sigma$ must be contained in the union of orbits of dimension $m-2$ or less. That is,

$$\Sigma \subset \bigcup_{0\leq j\leq m-2} M^j. \tag{15}$$

By the lower semicontinuity of the function $x \mapsto \dim Gx$, the set

$$M_+ = M^m \cup M^{m-1} \cup M^{m-2}$$

is open.

**Lemma 17.** *The restriction of $u$ to $M_+$ belongs to $W^{2,2}_{\mathrm{loc}}(M_+;N)$.*

*Proof.* Note that $\Sigma \cap M_+ = \Sigma \cap M^{m-2}$ is a smooth $(m-2)$-dimensional submanifold of $M_+$ consisting of $(m-2)$-dimensional orbits. Since $\theta$ is constant on the orbit $Gx$ whenever $x \in \Sigma$, Lemma 11 implies that there exists a closed set $\Sigma' \subset M_+$ with $\mathcal{H}^{m-2}(\Sigma') = 0$ such that $u \in W^{2,2}_{\mathrm{loc}}(M_+ \setminus \Sigma'; N)$.

Now for any $x \in M_+$, we have $\mathcal{H}^{m-2}(Gx) > 0$. Hence there exists an $a \in G$ such that $ax \notin \Sigma'$, and therefore, the open set $\lambda_a^{-1}(M_+ \backslash \Sigma')$ contains $x$. But by the equivariance of $u$, we have $u \in W^{2,2}_{\mathrm{loc}}(\lambda_a^{-1}(M_+ \backslash \Sigma'); N)$. This implies the claim. ☐

We want to prove that $\Sigma$ in fact consist of finitely many $(m-2)$-dimensional orbits only. To this end, we consider the $(m-2)$-dimensional Hausdorff measure and the elastica or Willmore energy of the orbits in question.

**Lemma 18.** *Let $x \in M$. Then there exists a neighbourhood $U \subset M$ of $x$ such that at most finitely many $(m-2)$-dimensional orbits intersect $U \cap \Sigma$.*

*Proof.* Assume, by way of contradiction, that there exists a sequence $(x_\ell)_{\ell \in \mathbb{N}}$ in $\Sigma$ such that $x_\ell \to x$ as $\ell \to \infty$ and $Gx_\ell \in Q^{m-2}$ for every $\ell \in \mathbb{N}$, but $Gx_\ell \neq Gx_{\ell'}$ for $\ell \neq \ell'$.

Every orbit $Gx_\ell$ is an $(m-2)$-dimensional submanifold of $M$ with $Gx_\ell \subset \Sigma$. Let $H_\ell$ denote the mean curvature of $Gx_\ell$. Let $R > 0$ be the injectivity radius of $x$ and let $r \in (0, R/2)$. Applying Lemma 16 to balls about $x_\ell$ with radius $r/2$, we conclude that there exists a constant $C_1$ (depending only on $M'$) such that

$$\left(8r^{2-m} + C_1 r^{4-m}\right) \mathcal{H}^{m-2}(Gx_\ell \cap B_r(x)) + \frac{1}{2} r^{4-m} \int_{Gx_\ell \cap B_r(x)} |H_\ell|^2 \, d\mathcal{H}^2 \geq 2$$

whenever $\ell$ is large enough so that $B_{r/2}(x_\ell) \subset B_r(x_0)$ and $r/2$ is less than the injectivity radius of $x_\ell$. On the other hand, we know that there exists a constant $C_2$ such that

$$\sum_{\ell=1}^{\infty} \left( \mathcal{H}^{m-2}(Gx_\ell) + \int_{Gx_\ell} |H_\ell|^2 \, d\mathcal{H}^{m-2} \right) \leq C_2 \liminf_{k \to \infty} \left( \bar{\tilde{\mu}}_{u_k}(M) + E_2(u_k) \right),$$

which is finite. This gives rise to the desired contradiction. ☐

Combining Lemma 18 with (15), we see that for each $x \in M^0 \cup M^{m-3}$, there exists a neighbourhood $U$ of $x$ such that $U \cap \Sigma \subset M^0 \cup M^{m-3}$. But under the assumptions of Theorem 2, we have $\mathcal{H}^{m-2}\left(M^0 \cup M^{m-3}\right) = 0$. Therefore, $M^0 \cup M^{m-3}$ cannot support a nontrivial $(m-2)$-dimensional varifold. Thus $U \cap \Sigma = \emptyset$. It follows that the convergence $du_k \to du$ is strong in $L^2(U)$ and therefore $u \in K_{u_0}(U; N)$.

Taking the union of all such $U$, we conclude that there exists an open set $M_- \subset M$ such that $M^0 \cup M^{m-3} \subset M_-$ and $u \in K_{u_0}(M_-; N)$. On the other hand, Lemma 17 implies that $u \in K_{u_0}(M_+; N)$. Since $M \subset M_- \cup M_+$, we conclude that $u \in K_{u_0}(M; N)$. Hence $u \in \mathcal{A}_{u_0}$. Since

$$E_2(u) \le \liminf_{k\to\infty} E_2(u_k) = \inf_{\mathcal{A}_{u_0}} E_2,$$

this completes the proof of Theorem 2.

For the proof of Theorem 5, we use Lemma 17 again. If $m = 3$, then the hypothesis of the theorem implies $M = M_+$, so in this case $u \in W^{2,2}(M; N)$. According to Proposition 14, it is then a weakly biharmonic map, and it follows that it is very weakly biharmonic as well. If $m = 4$ and $Q^0$ is finite and $Q^1 = \emptyset$, then $M \setminus M_+ = M^0$ is finite, so it follows from Lemma 17 and from Lemma 15 that $u$ is very weakly biharmonic.

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